MATH2050C Assignment 11

Deadline: April 9, 2018.

Hand in: 5.3 no. 5, 6, 12; Suppl. Ex. no. 1.

Section 5.3 no. 2, 3, 4, 5, 6, 12, 13.

Supplementary Exercise

1. Let f be continuous on $[a,b]$. For $x_1, x_2, \cdots, x_n \in [a,b]$, show that there is some $\xi \in [a,b]$ such that
$$f(\xi) = \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)).$$
Is the conclusion still valid when $[a,b]$ is replaced by (a,b)?

2. Let h be an increasing function on some interval $[a,b]$, that is, $h(x) \le h(y)$ for $x \le y$.
   (a) Show that $\lim_{x\to x_0^+} h(x)$ and $\lim_{x\to x_0^-} h(x)$ always exist for every $x_0 \in (a,b)$.
   (b) Show that f is continuous at $x_0 \in (a,b)$ if and only if the right hand limit is equal to the left hand limit at x_0.
   (c) Show that h is continuous on $[a,b]$ if and only if the range of h is $[h(a), h(b)]$.
   (d) Optional. Show that if for a given number $k > 0$, the set $\{x \in (a,b) : \lim_{x\to z^+} h(x) - \lim_{x\to z^-} h(x) \ge k\}$ is a finite set.
   (e) Optional. Deduce from (d) that h has at most countably many points of discontinuity.

Existence of Zeros for Continuous Functions

We give a proof of the theorem on the existence of zeros different from the bisection method. Proposition 1 will be used many times in this course and 2060.

Proposition 1. Let f be defined on (a, b) and continuous at $x_0 \in (a, b)$. When $\alpha = f(x_0) > 0$, there is some small $\delta > 0$ such that $f(x) > \alpha/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. When $f(x_0) < 0$, there is some small $\delta > 0$ such that $f(x) < f(x_0)/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Proof. Let $\varepsilon = \alpha/2$. There exists some $\delta > 0$ such that $|f(x) - f(x_0)| = |f(x) - \alpha| < \alpha/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. (We could choose δ so small that this interval is included in (a, b).) It follows that $f(x) - \alpha > -\alpha/2$, that is, $f(x) > \alpha/2$, on this interval.

Here are some remarks.

- First, by choosing a smaller δ, $(x_0 - \delta, x_0 + \delta)$ could be replaced by $[x_0 - \delta, x_0 + \delta]$. Second, it means in particular that $f > 0$ on $(x_0 - \delta, x_0 + \delta)$.
- When f is right hand continuous at $x_0 \in [a, b)$ or left hand continuous, the conclusion holds on $[x_0, x_0 + \delta]$ or $[x_0 - \delta, x_0]$ repectively.
- Similar results hold when $f(x_0) < 0$. Simply consider $-f$.

Theorem 2. Let f be continuous on $[a, b]$ satisfying $f(a)f(b) < 0$. There exists some $c \in (a, b)$ such that $f(c) = 0$.

Proof. Without loss of generality assume $f(a) < 0$ and $f(b) > 0$. Consider the set $E = \{c \in [a, b] : f > 0 \text{ on } [a, c]\}$. By Proposition 1, f is positive on $[a, a + \delta]$ for some small δ. Hence E is nonempty by taking $c = a + \delta$. On the other hand, E is bounded above by b. By Order-Completeness Property, $\xi = \sup E \leq b$ exists. By the definition of supremum, we can find a sequence $z_n \in E$ such that $\lim_{n\to\infty} z_n = \xi$. By continuity, $0 \geq \lim_{n\to\infty} f(z_n) = f(\xi)$. On the other hand, if $f(\xi) < 0$, Proposition 1 asserts that $f(x) < 0$ for $x \in [\xi - \delta_1, \xi + \delta_1]$ for some small δ_1, thus $f < 0$ on $[a, \xi + \delta_1]$, contradicting the fact that ξ is the supremum of E. One must have $f(\xi) = 0$.